

CONSTRUCTION OF NERON DESINGULARIZATION FOR TWO DIMENSIONAL RINGS

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ABSTRACT. An algorithmic proof of the General Neron Desingularization theorem is given for 2-dimensional local rings and morphisms with small singular locus.

Key words : Smooth morphisms, regular morphisms

2010 Mathematics Subject Classification: Primary 13B40, Secondary 14B25, 13H05, 13J15.

1. INTRODUCTION

The General Neron Desingularization Theorem, first proved by the second author has many important applications. One application is the generalization of Artin's famous approximation theorem (Artin [2], Popescu [9], [10]).

Theorem 1. (*General Neron Desingularization, André [1], Popescu [8], [9], [10], Swan [11]*) *Let $u : A \rightarrow A'$ be a regular morphism of Noetherian rings and B an A -algebra of finite type. Then any A -morphism $v : B \rightarrow A'$ factors through a smooth A -algebra C , that is v is a composite A -morphism $B \rightarrow C \rightarrow A'$.*

The proof of this theorem is not constructive. Constructive proofs for one-dimensional rings were given in A. Popescu, D. Popescu [7], and Pfister, Popescu [6]. In this paper we will treat the 2-dimensional case.

2. CONSTRUCTIVE GENERAL NERON DESINGULARIZATION IN A SPECIAL CASE

Let $u : A \rightarrow A'$ be a flat morphism of Noetherian Cohen-Macaulay local rings of dimension 2. Suppose that the maximal ideal \mathfrak{m} of A generates the maximal ideal of A' and the completions of A, A' are isomorphic. Moreover suppose that A' is Henselian, and u is a regular morphism.

Let $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$. If $f = (f_1, \dots, f_r)$, $r \leq n$ is a system of polynomials from I then we can define the ideal Δ_f generated by all $r \times r$ -minors of the Jacobian matrix $(\partial f_i / \partial Y_j)$. After Elkik [4] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I) \Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. Then B_P , $P \in \text{Spec } B$ is essentially smooth over A if and only if $P \not\supset H_{B/A}$ by the Jacobian criterion for smoothness. Thus $H_{B/A}$ measures the non smooth locus of B over A . B is *standard smooth* over A if there exists f in I as above such that $B = ((f) : I) \Delta_f B$.

The aim of this paper is to give an easy algorithmic proof of the following theorem.

Theorem 2. *Any A -morphism $v : B \rightarrow A'$ such that $v(H_{B/A})A'$ is $\mathfrak{m}A'$ -primary factors through a standard smooth A -algebra B' .*

Proof. We choose $\gamma, \gamma' \in v(H_{B/A})A' \cap A$ such that γ, γ' is a regular sequence in A , let us say $\gamma = \sum_{i=1}^q v(b_i)z_i$, $\gamma' = \sum_{i=1}^q v(b_i)z'_i$ for some $b_i \in H_{B/A}$ and $z_i, z'_i \in A'$. Set $B_0 = B[Z, Z']/(f, \tilde{f})$, where $f = -\gamma + \sum_{i=1}^q b_i Z_i \in B[Z]$, $Z = (Z_1, \dots, Z_q)$, $\tilde{f} = -\gamma' + \sum_{i=1}^q b_i Z'_i \in B[Z']$, $Z' = (Z'_1, \dots, Z'_q)$ and let $v_0 : B_0 \rightarrow A'$ be the map of B -algebras given by $Z \rightarrow z$, $Z' \rightarrow z'$. Changing B by B_0 we may suppose that $\gamma, \gamma' \in H_{B/A}$.

We need the following lemmas.

- Lemma 3.** (1) ([8, Lemma 3.4]) *Let B_1 be the symmetric algebra $S_B(I/I^2)$ of I/I^2 over B . Then $H_{B/A}B_1 \subset H_{B_1/A}$ and $(\Omega_{B_1/A})_\gamma$ is free over $(B_1)_\gamma$ for any $\gamma \in H_{B/A}$.*
- (2) ([11, Proposition 4.6]) *Suppose that $(\Omega_{B/A})_\gamma$ is free over B_γ . Let $I' = (I, Y') \subset A[Y, Y']$, $Y' = (Y'_1, \dots, Y'_n)$. Then $(I'/I'^2)_\gamma$ is free over B_γ .*
- (3) ([10, Corollary 5.10]) *Suppose that $(I/I^2)_\gamma$ is free over B_γ . Then a power of γ is in $((g) : I)\Delta_g$ for some $g = (g_1, \dots, g_r)$, $r \leq n$ in I .*

Using (1) of Lemma 3 we can reduce the proof to the case when $\Omega_{B_\gamma/A}$ and $\Omega_{B_{\gamma'}/A}$ are free over B_γ respectively $B_{\gamma'}$. Let B_1 be given by (1) of Lemma 3. The inclusion $B \subset B_1$ has a retraction w which maps I/I^2 to zero. For the reduction we change B, v by B_1, vw .

Using (2) from Lemma 3 we may reduce the proof to the case when $(I/I^2)_\gamma$ (resp. $(I/I^2)_{\gamma'}$) is free over B_γ (resp. $B_{\gamma'}$). Indeed, since $\Omega_{B_\gamma/A}$ is free over B_γ we see that changing I with $(I, Y') \subset A[Y, Y']$ we may suppose that $(I/I^2)_\gamma$ is free over B_γ . Similarly, for γ' .

Using (3) from Lemma 3 we may reduce the proof to the case when a power of γ (resp. γ') is in $((f) : I)\Delta_f$ (resp. $((f') : I)\Delta_{f'}$) for some $f = (f_1, \dots, f_r)$, $r \leq n$ and $f' = (f'_1, \dots, f'_{r'})$, $r' \leq n$ from I .

We may now assume that a power d (resp. d') of γ (resp. γ') has the form $d \equiv P = \sum_{i=1}^q M_i L_i$ modulo I , $d' \equiv P' = \sum_{i=1}^{q'} M'_i L'_i$ modulo I for some $r \times r$ (resp. $r' \times r'$) minors M_i (resp. M'_i) of $(\partial f / \partial Y)$ (resp. $(\partial f' / \partial Y)$) and $L_i \in ((f) : I)$ (resp. $L'_i \in ((f') : I)$).

The Jacobian matrix $(\partial f / \partial Y)$ (resp. $(\partial f' / \partial Y)$) can be completed with $(n - r)$ (resp. $(n - r')$) rows from A^n obtaining a square n matrix H_i (resp. H'_i) such that $\det H_i = M_i$ (resp. $\det H'_i = M'_i$). This is easy using just the integers 0, 1.

Let $\bar{A} = A/(d^3)$, $\bar{B} = \bar{A} \otimes_A B$, $\bar{A}' = A'/(d'^3 A')$, $\bar{v} = \bar{A} \otimes_A v$.

We will now construct a standard smooth A -algebra D and an A -morphism $\omega : D \rightarrow A'$ such that $y = v(Y) \in \text{Im } \omega + d^3 A'$.

Lemma 4. *There exists a standard smooth A -algebra D such that \bar{v} factors through $\bar{D} = \bar{A} \otimes_A D$.*

¹For the algorithm we have to choose γ, γ' more carefully: $\gamma \equiv \sum_{i=1}^q b_i(y')z_i$ modulo (γ^t, γ'^t) , $\gamma' \equiv \sum_{i=1}^q b_i(y')z'_i$ modulo (γ^t, γ'^t) with $z_i, z'_i \in A$, and $y'_i \equiv v(Y_i)$ modulo \mathfrak{m}^N in A , $N \gg 0$.

²Let M be a finitely represented B -module and $B^m \xrightarrow{(a_{ij})} B^n \rightarrow M \rightarrow 0$ a presentation then $S_B(M) = B[T_1, \dots, T_n]/J$ with $J = (\{\sum_{i=1}^n a_{ij} T_i\}_{j=1, \dots, m})$.

Proof. Let $y' \in A^n$ be such that $y = v(Y) \equiv y'$ modulo $(d^3, d'^3)A'$, let us say $y - y' \equiv d'^2\varepsilon$ modulo d^3 for $\varepsilon \in d'A'^n$. Thus $I(y') \equiv 0$ modulo $(d^3, d'^3)A'$.

Recall that we have $d' \equiv P'$ modulo I and so $P'(y') \equiv d'$ modulo (d^3, d'^3) in A . Thus $P'(y') \equiv d's$ modulo d^3 for a certain $s \in A$ with $s \equiv 1$ modulo d' .

Let G'_i be the adjoint matrix of H'_i and $G_i = L_i G'_i$. We have $G_i H'_i = H'_i G_i = M'_i L'_i \text{Id}_n$ and so $P'(y') \text{Id}_n = \sum_{i=1}^{q'} G_i(y') H'_i(y')$.

But H'_i is the matrix $(\partial f'_k / \partial Y_j)_{k \in [r'], j \in [n]}$ completed with some $(n - r')$ rows of 0, 1. Especially we obtain

$$(1) \quad (\partial f' / \partial Y) G_i = M'_i L'_i (\text{Id}_{r'} | 0).$$

Then $t_i := H'_i(y')\varepsilon \in d'A'^n$ satisfies

$$G_i(y')t_i = M'_i(y')L'_i(y')\varepsilon$$

and so

$$\sum_{i=1}^q G_i(y')t_i = P'(y')\varepsilon \equiv d's\varepsilon \text{ modulo } d^3.$$

It follows that

$$s(y - y') \equiv d' \sum_{i=1}^{q'} G_i(y')t_i \text{ modulo } d^3.$$

Note that $t_{ij} = t_{i1}$ for all $i \in [r']$ and $j \in [n]$ because the first r' rows of H'_i does not depend on i (they are the rows of $(\partial f' / \partial Y)$).

Let

$$(2) \quad h = s(Y - y') - d' \sum_{i=1}^{q'} G_i(y')T_i,$$

where $T_i = (T_1, \dots, T_{r'}, T_{i,r'+1}, \dots, T_{i,n})$, $i \in [q']$ are new variables. We will use also $T_{ij} = T_i$ for $i \in [r']$, $j \in [n]$ because it is convenient sometimes. The kernel of the map $\bar{\varphi} : \bar{A}[Y, T] \rightarrow \bar{A}'$ given by $Y \rightarrow y$, $T \rightarrow t$ contains h modulo d^3 . Since

$$s(Y - y') \equiv d' \sum_{i=1}^{q'} G_i(y')T_i \text{ modulo } h$$

and

$$f'(Y) - f'(y') \equiv \sum_j (\partial f' / \partial Y_j)(y')(Y_j - y'_j)$$

modulo higher order terms in $Y_j - y'_j$, by Taylor's formula we see that for $p' = \max_i \deg f'_i$ we have

$$(3) \quad s^{p'} f'(Y) - s^{p'} f'(y') \equiv \sum_j s^{p'-1} d' (\partial f' / \partial Y_j)(y') \sum_{i=1}^{q'} G_{ij}(y') T_{ij} + d'^2 Q$$

modulo h where $Q \in T^2 A[T]^{r'}$. We have $f'(y') \equiv d'^2 b'$ modulo d^3 for some $b' \in d'A'^{r'}$. Then

$$(4) \quad g_i = s^{p'} b'_i + s^{p'} T_i + Q_i, \quad i \in [r']$$

modulo d^3 is in the kernel of $\bar{\varphi}$. Indeed, we have $s^{p'} f'_i = d'^2 g_i$ modulo (h, d^3) because of (3). Thus $d'^2 \bar{\varphi}(g) = d'^2 g(t) \in (h(y, t), f'(y)) \in d^3 A'$ and so $g(t) \in d^3 A'$, because u is flat and d' is regular on $A/(d^3)$. Set $E = \bar{A}[Y, T]/(I, g, h)$ and let $\bar{\psi} : E \rightarrow \bar{A}'$ be the map induced by $\bar{\varphi}$. Clearly, \bar{v} factors through $\bar{\psi}$ because \bar{v} is the composed map $\bar{B} = \bar{A}[Y]/I \rightarrow E \xrightarrow{\bar{\psi}} \bar{A}'$.

Now we will see that there exist $s', s'' \in E$ such that $E_{ss's''}$ is smooth over \bar{A} and $\bar{\psi}$ factors through $E_{ss's''}$.

Note that the $r' \times r'$ -minor s' of $(\partial g / \partial T)$ given by the first r' -variables T is from $s^{r'} p' + (T) \subset 1 + (d', T)$ because $Q \in (T)^2$. Then $V = (\bar{A}[Y, T]/(h, g))_{ss'}$ is smooth over \bar{A} . As in [6] we claim that $I\bar{A}[Y, T] \subset (h, g)\bar{A}[Y, T]_{ss's''}$ for some $s'' \in 1 + (d', d^3, T)A[Y, T]$. Indeed, we have $P'I\bar{A}[Y, T] \subset (f')A[Y, T] \subset (h, g)\bar{A}[Y, T]_s$ and so $P'(y' + s^{-1}d'G(y')T)I \subset (h, g, d^3)A[Y, T]_s$. Since $P'(y' + s^{-1}d'G(y')T) \in P'(y') + d'(T)V$ we get $P'(y' + s^{-1}d'G(y')T) \equiv d's''$ modulo d^3 for some $s'' \in 1 + (T)A[Y, T]$. It follows that $s''I \subset ((h, g) : d'), d^3)A[Y, T]_{ss'}$. Thus $s''I$ is contained modulo d^3 in $(0 :_V d') = 0$ because d' is regular on V , the map $\bar{A} \rightarrow V$ being flat. This shows our claim. It follows that $I \subset (d^3, h, g)A[Y, T]_{ss's''}$. Thus $E_{ss's''} \cong V_{s''}$ is a \bar{B} -algebra which is also standard smooth over \bar{A} .

As $u(s) \equiv 1$ modulo d' and $\bar{\psi}(s'), \bar{\psi}(s'') \equiv 1$ modulo (d', d^3, t) , $d, d', t \in \mathfrak{m}_{A'}$ we see that $u(s), \bar{\psi}(s'), \bar{\psi}(s'')$ are invertible because A' is local. Thus $\bar{\psi}$ (and so \bar{v}) factors through the standard smooth \bar{A} -algebra $E_{ss's''}$, let us say by $\bar{\omega} : E_{ss's''} \rightarrow \bar{A}'$.

Now, let $Y' = (Y'_1, \dots, Y'_n)$, and D be the A -algebra isomorphic with $(A[Y, T]/(I, h, g))_{ss's''}$ by $Y' \rightarrow Y, T \rightarrow T$. Since A' is Henselian we may lift $\bar{\omega}$ to a map $(A[Y, T]/(I, h, g))_{ss's''} \rightarrow A'$ which will correspond to a map $\omega : D \rightarrow A'$. Then \bar{v} factors through \bar{D} , let us say $\bar{B} \rightarrow \bar{D} \rightarrow \bar{A}'$, where the first map is given by $Y \rightarrow Y'$. Note that v does not factor through D . \square

Let $\delta : B \otimes_A D \cong D[Y]/ID[Y] \rightarrow A'$ be the A -morphism given by $b \otimes \lambda \rightarrow v(b)\omega(\lambda)$.

Claim: δ factors through a special finite type $B \otimes_A D$ -algebra \tilde{E} .

The proof will follow the proof of Lemma 4. Note that the map $\bar{B} \rightarrow \bar{D}$ is given by $Y \rightarrow Y' + d^3 D$. Thus $I(Y') \equiv 0$ modulo $d^3 D$. Set $\tilde{y} = \omega(Y')$. Since \bar{v} factors through $\bar{\omega}$ we get $y - \tilde{y} = v(Y) - \tilde{y} \in d^3 A^n$, let us say $y - \tilde{y} = d^2 \nu$ for $\nu \in dA^n$.

Recall that $P = \sum_i L_i \det H_i$ for $L_i \in ((f) : I)$. We have $d \equiv P$ modulo I and so $P(Y') \equiv d$ modulo d^3 in D because $I(Y') \equiv 0$ modulo $d^3 D$. Thus $P(Y') = d\tilde{s}$ for a certain $\tilde{s} \in D$ with $\tilde{s} \equiv 1$ modulo d . Let \tilde{G}'_i be the adjoint matrix of H_i and $\tilde{G}_i = L_i \tilde{G}'_i$. We have $\sum_i \tilde{G}_i H_i = \sum_i H_i \tilde{G}_i = P \text{Id}_n$ and so

$$d\tilde{s} \text{Id}_n = P(Y') \text{Id}_n = \sum_i \tilde{G}_i(Y') H_i(Y').$$

But H_i is the matrix $(\partial f_i / \partial Y_j)_{i \in [r], j \in [n]}$ completed with some $(n - r)$ rows from 0, 1. Especially we obtain

$$(5) \quad (\partial f / \partial Y) \sum_i \tilde{G}_i = (P \text{Id}_r | 0).$$

Then $\tilde{t}_i := \omega(H_i(Y'))\nu \in dA^m$ satisfies

$$\sum_i \tilde{G}_i(Y')\tilde{t}_i = P(Y')\nu = d\tilde{s}\nu$$

and so

$$\tilde{s}(y - \tilde{y}) = d \sum_i \omega(\tilde{G}_i(Y'))\tilde{t}_i.$$

Let

$$(6) \quad \tilde{h} = \tilde{s}(Y - Y') - d \sum_i \tilde{G}_i(Y')\tilde{T}_i,$$

where $\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_n)$ are new variables. The kernel of the map $\tilde{\varphi} : D[Y, \tilde{T}] \rightarrow A'$ given by $Y \rightarrow y, \tilde{T} \rightarrow \tilde{t}$ contains \tilde{h} . Since

$$\tilde{s}(Y - Y') \equiv d \sum_i \tilde{G}_i(Y')\tilde{T}_i \text{ modulo } \tilde{h}$$

and

$$f(Y) - f(Y') \equiv \sum_j (\partial f / \partial Y_j)((Y')(Y_j - Y'_j))$$

modulo higher order terms in $Y_j - Y'_j$, by Taylor's formula we see that for $p = \max_i \deg f_i$ we have

$$(7) \quad \tilde{s}^p f(Y) - \tilde{s}^p f(Y') \equiv \sum_j \tilde{s}^{p-1} d(\partial f / \partial Y_j)(Y') \sum_i \tilde{G}_{ij}(Y')\tilde{T}_{ij} + d^2 \tilde{Q}$$

modulo \tilde{h} where $\tilde{Q} \in \tilde{T}^2 D[\tilde{T}]^r$. We have $f(Y') = d^2 \tilde{b}$ for some $\tilde{b} \in dD^r$. Then

$$(8) \quad \tilde{g}_i = \tilde{s}^p \tilde{b}_i + \tilde{s}^p \tilde{T}_i + \tilde{Q}_i, \quad i \in [r]$$

is in the kernel of $\tilde{\varphi}$. Indeed, we have $\tilde{s}^p f_i = d^2 \tilde{g}_i$ modulo \tilde{h} because of (7) and $P(Y') = d\tilde{s}$. Thus $d^2 \varphi(\tilde{g}) = d^2 \tilde{g}(t) \in (\tilde{h}(y, \tilde{t}), f(y)) = (0)$ and so $\tilde{g}(\tilde{t}) = 0$. Set $\tilde{E} = D[Y, \tilde{T}]/(I, \tilde{g}, \tilde{h})$ and let $\tilde{\psi} : \tilde{E} \rightarrow A'$ be the map induced by $\tilde{\varphi}$. Clearly, v factors through $\tilde{\psi}$ because v is the composed map $B \rightarrow B \otimes_A D \cong D[Y]/I \rightarrow \tilde{E} \xrightarrow{\tilde{\psi}} A'$.

Finally we will prove that there exist $\tilde{s}', \tilde{s}'' \in \tilde{E}$ such that $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$ is standard smooth over A and $\tilde{\psi}$ factors through $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$.

Note that the $r \times r$ -minor \tilde{s}' of $(\partial \tilde{g} / \partial \tilde{T})$ given by the first r -variables \tilde{T} is from $\tilde{s}^{rp} + (\tilde{T}) \subset 1 + (d, \tilde{T})$ because $\tilde{Q} \in (\tilde{T})^2$. Then $\tilde{V} = (D[Y, \tilde{T}]/(\tilde{h}, \tilde{g}))_{\tilde{s}\tilde{s}'}$ is smooth over D . We claim that $I \subset (\tilde{h}, \tilde{g})D[Y, \tilde{T}]_{\tilde{s}\tilde{s}'\tilde{s}''}$ for some other $\tilde{s}'' \in 1 + (d, \tilde{T})D[Y, \tilde{T}]$. Indeed, we have $PID[Y] \subset (f)D[Y] \subset (\tilde{h}, \tilde{g})D[Y, \tilde{T}]_{\tilde{s}}$ and so $P(Y' + \tilde{s}^{-1}d \sum_i \tilde{G}_i(Y')\tilde{T}_i)I \subset (\tilde{h}, \tilde{g})D[Y, \tilde{T}]_{\tilde{s}}$. Since $P(Y' + \tilde{s}^{-1}d \sum_i \tilde{G}_i(Y')\tilde{T}_i) \in P(Y') + d(\tilde{T})$ we get $P(Y' + \tilde{s}^{-1}d \sum_i \tilde{G}_i(Y')\tilde{T}_i) = d\tilde{s}''$ for some $\tilde{s}'' \in 1 + (\tilde{T})D[Y, \tilde{T}]$. It follows that $\tilde{s}''I \subset ((\tilde{h}, \tilde{g}) : d)D[Y, \tilde{T}]_{\tilde{s}\tilde{s}'}$. Thus $\tilde{s}''I \subset (0 :_{\tilde{V}} d) = 0$, which shows our claim. It follows that $I \subset (\tilde{h}, \tilde{g})D[Y, \tilde{T}]_{\tilde{s}\tilde{s}'\tilde{s}''}$. Thus $\tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''} \cong \tilde{V}_{\tilde{s}''}$ is a B -algebra which is also standard smooth over D and A .

As $\omega(\tilde{s}) \equiv 1$ modulo d and $\tilde{\psi}(\tilde{s}'), \tilde{\psi}(\tilde{s}'') \equiv 1$ modulo (d, \tilde{t}) , $d, \tilde{t} \in \mathfrak{m}A'$ we see that $\omega(\tilde{s}), \tilde{\psi}(\tilde{s}'), \tilde{\psi}(\tilde{s}'')$ are invertible because A' is local. Thus $\tilde{\psi}$ (and so v) factors through the standard smooth A -algebra $B' = \tilde{E}_{\tilde{s}\tilde{s}'\tilde{s}''}$. \square

3. THE ALGORITHM

We obtain the following algorithm (which will be implemented in SINGULAR as a library and available 2017).

Algorithm Neron Desingularization

Input: $N \in \mathbb{Z}_{>0}$ a bound

$A := k[x]_{(x)}/J, J = (h_1, \dots, h_p) \subseteq k[x], x = (x_1, \dots, x_t), k$, a field

$B := A[Y]/I, I = (g_1, \dots, g_l) \subseteq k[x, Y], Y = (Y_1, \dots, Y_n)$

$v : B \rightarrow A' \subseteq K[[x]]/JK[[x]]$ an A -morphism, given by $y' = (y'_1, \dots, y'_n) \in k[x]^n$, approximations $\text{mod}(x)^N$ of $v(Y)$.

Output: A Neron desingularization of $v : B \rightarrow A'$ or the message "the bound is too small"

1. Compute $H_{B/A} = (b_1, \dots, b_q)_B$ and $H_{B/A} \cap A$.
2. if $\dim A/H_{B/A} \cap A = 0$ choose $\gamma, \gamma' \in H_{B/A} \cap A$, a regular sequence in A and go to 6.
3. choose $\gamma, \gamma' \in H_{B/A}(y')$, a regular sequence in A .
4. Write $\gamma \equiv \sum_{i=1}^q b_i(y')y'_{i+n}$ modulo (γ^t, γ'^t) , $\gamma' \equiv \sum_{i=1}^q b_i(y')y'_{i+n+q}$ modulo (γ^t, γ'^t) for some t and $y_j \in k[x]$.
5. $g_{l+1} := -\gamma + \sum_{i=1}^q b_i Y_{i+n}$, $g_{l+2} := -\gamma' + \sum_{i=1}^q b_i Y_{i+n+q}$, $Y = (Y_1, \dots, Y_{n+2q})$, $y' = (y'_1, \dots, y'_{n+2q})$, $I = (g_1, \dots, g_{l+2})$; $l := l + 2$; $n := n + 2q$; $B = A[Y]/I$.
6. $B := S_B(I/I^2)$, v trivially extended. Write $B := A[Y]/I$, $n := |Y|$, $Y := Y, Z, Z = (Z_1, \dots, Z_n)$, $I := (I, Z)$, $B := A[Y]/I$, v trivially extended.
7. Compute $f = (f_1, \dots, f_r)$, and $f' = (f'_1, \dots, f'_{r'})$ such that a power d of γ , resp. d' of γ' is in $((f) : I)\Delta_f$, resp. in $((f') : I)\Delta'_f$.
if $(d^3, d'^3) \not\supseteq (x)^N$ return to "the bound is too small".
8. Choose r -minors M_i (resp. r' -minors M'_i) of $(\partial f/\partial Y)$, (resp. $(\partial f'/\partial Y)$) and $L_i \in ((f) : I)$, (resp. $L'_i \in ((f') : I)$) such that for $P = \sum_i M_i L_i$ (resp. $P' = \sum_i M'_i L'_i$), $d \equiv P$ modulo I (resp. $d' \equiv P'$ modulo I).
9. Complete the Jacobian matrix $(\partial f/\partial Y)$ (resp. $(\partial f'/\partial Y)$) by $(n - r)$ (resp. $(n - r')$) rows of 0, 1 to obtain square matrices H_i (resp. H'_i) such that $\det H_i = M_i$ (resp. $\det H'_i = M'_i$).
10. Write $P'(y') = d's$ modulo d^3 for $s \in A$, $s \equiv 1$ modulo d' .
11. For $i = 1$ to q' compute G'_i the adjoint matrix of H'_i and $G_i = L_i G'_i$.
12. $h := s(Y - y') - d' \sum_i G_i(y')T_i$, $T_i = (T_1, \dots, T_{r'}, T_{i,r'+1}, \dots, T_{i,n})$.
13. $p' := \max_i \{\deg f'_i\}$ write $s^{p'} f'(Y) - s^{p'} f'(y') = \sum_j s^{p'-1} d' \partial f'/\partial Y(y') \sum_i G_{ij}(y') T_{ij} + d'^2 Q$ modulo h and $f'(y') = d'^2 b'$ modulo d^3 . For $i = 1$ to r' $g_i := s^{p'} b'_i + s^{p'} T_i + Q_i$.
14. Compute s' the r' -minor of $(\partial g/\partial T)$ given by the first r' variables and s'' such that $P(y' + s^{-1} d' \sum_i G_i(y') T_i) = d' s''$ modulo d^3 .
15. $D := (A[Y', T]/(I, g, h))_{ss's''}$, $Y' = (Y'_1, \dots, Y'_n)$, $g := g(Y')$, $I := I(Y')$, $h := h(Y')$.

Write $P(Y') = d\tilde{s}$, $\tilde{s} \equiv 1$ modulo d .

16. Compute \tilde{G}'_i the adjoint matrix of H_i and $\tilde{G}_i = L_i \tilde{G}'_i$.
17. $\tilde{h} := \tilde{s}(Y - Y') - d \sum_{i=1}^q \tilde{G}_i \tilde{T}_i$, $\tilde{T}_i = (\tilde{T}_1, \dots, \tilde{T}_r, \tilde{T}_{i,r+1}, \dots, \tilde{T}_{i,n})$.
18. $p := \max_i \{\deg f_i\}$
- Write
- $\tilde{s}^p f(Y) - \tilde{s}^p f(Y') = \sum_j \tilde{s}^{p-1} d \partial f / \partial Y(Y') \sum_i \tilde{G}_{ij}(Y') \tilde{T}_{ij} + d'^2 \tilde{Q}$ modulo \tilde{h} and $f(Y') = d^2 \tilde{b}$, $\tilde{b} \in dD^r$.
19. For $i = 1$ to r , $\tilde{g}_i := \tilde{s}^p \tilde{b}_i + \tilde{s}^p \tilde{T}_i + \tilde{Q}_i$.
20. Compute \tilde{s}' the $r \times r$ -minors of $(\partial \tilde{g} / \partial \tilde{T})$ given by the first r variables of \tilde{T} . Compute \tilde{s}'' such that $P(Y' + \tilde{s}^{-1} d \sum_i \tilde{G}_i(Y') \tilde{T}) = d \tilde{s}''$.
21. return $D[Y, \tilde{T} / (I, \tilde{g}, \tilde{h})_{\tilde{s} \tilde{s}' \tilde{s}''}$.

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